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A Class of Quasilinear Parabolic Equations with Infinite Delay and Application to a Problem of Viscoelasticity

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A semigroup approach to differential-delay equations is developed which reduces such equations to ordinary differential equations on a Banach space of histories and seems more suitable for certain partial integro-differential equations than the standard theory. The method is applied to prove a local-time existence theorem for equations of the form $u_{tt} = g(u_{xt}, u_x^t)_x$, where $\partial g/\partial u_{xt} > 0$. On a formal level, it is demonstrated that the stretching of filaments of viscoelastic liquids can be described by an equation of this form.

1. INTRODUCTION

Consider a linear functional differential equation of the form

$$\dot{y} = Ly^t, \tag{1.1}$$

where y^t denotes the history of y on the interval $(-\infty, t]$: $y^t(s) = y(t+s)$, and L is a continuous linear operator from $C_b^{\text{lim}}(-\infty, 0]$ into $\mathbb{R}$. Here $C_b^{\text{lim}}(-\infty, 0]$ is the space of bounded continuous functions $(-\infty, 0] \to \mathbb{R}$, which converge to a limit as $t \to -\infty$. Equations of this kind can be regarded as evolution problems on $C_b^{\text{lim}}(-\infty, 0]$ and can be studied using semigroup theory. A generator A of a C^0-semigroup can be associated to (1.1) as follows (see [2]):

$$\begin{aligned} A\psi(s) &= \frac{d\psi(s)}{ds} \qquad \text{for} \quad -\infty < s < 0 \\ &= L\psi \qquad \text{for} \quad s = 0 \end{aligned} \tag{1.2}$$

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for any $\psi \in C_b^{\lim}(-\infty, 0]$ such that $d\psi/ds \in C_b^{\lim}(-\infty, 0]$ and $d\psi(s=0)/ds = L\psi$. However, if partial differential equations are considered, the operator A has some inconvenient properties. For instance, the semigroup generated by A is not analytic even for $L=0$. This is disturbing if one studies partial integro-differential equations such as the following:

$$u_t(x, t) = u_{xx}(x, t) + \int_{-\infty}^{t} \alpha(t-s) f(u(x, s), u_x(x, s))\, ds \tag{1.3}$$

with, e.g., Dirichlet or Neumann boundary conditions. One is tempted to regard (1.3) as a perturbation of the heat equation and apply the results of Sobolevskii [10] to prove existence theorems. If, however, one follows the recipe outlined above, then even the heat equation itself is no longer associated to an analytic semigroup.

Another method of dealing with differential-delay equations is a direct approach without using the idea that a functional-differential equation can be represented as an ordinary differential equation on a space of histories. Such an approach is developed in [12, 14] for finite-dimensional problems and in [13] for semilinear parabolic partial differential equations. In these papers, a general class of admissible history spaces is defined by abstract conditions.

Here, we shall consider a representation of equations like (1.1) as evolution problems on the space $C_b^{\lim}$; this representation will, however, be different from (1.2). At first sight, our method seems to impose rather restrictive conditions on the given history and does not seem to admit histories in the general spaces considered in [12–14], even if the functionals are defined on such a space. However, this is only apparent, and we shall show in Remark 2 at the end of this Introduction, how one can get around this problem.

Let F be a smooth (possibly nonlinear) functional from $C_b^{\lim}(-\infty, 0]$ into $\mathbb{R}$. Moreover, for every function $\psi \in C_b^{\lim}(-\infty, 0]$ we define $T_r \psi \in C_b^{\lim}(-\infty, 0]$ by $T_r \psi(s) = \psi(s+r)$ for each $r \leqslant 0$. Then with F we associate the following operator $\hat{F}: C_b^{\lim}(-\infty, 0] \to C_b^{\lim}(-\infty, 0]$:

$$\hat{F}(\psi)(s) = F(T_s \psi) \tag{1.4}$$

(or, if F is allowed to depend on t, $\hat{F}(\psi, t)(s) = F(T_s \psi, t+s)$). There is some temptation to rewrite the equation $\dot{y} = F(y^t)$ in the abstract form $\dot{\hat{y}} = \hat{F}(\hat{y})$, $\hat{y} \in C_b^{\lim}(-\infty, 0]$. However, this is incorrect. The reason is that functional differential equations of this kind are usually associated with an initial condition

$$y(t) = y_0(t) \qquad \text{for} \quad t \leqslant 0, \tag{1.5}$$

and y_0 is not required to satisfy the equation. We can, however, define $f(t)$ by

$$
\begin{aligned}
f(t) &= \frac{\partial y_0(t)}{\partial t} - F(y_0^t) \qquad \text{for} \quad t \leqslant 0 \\
&= 0 \qquad\qquad\qquad\qquad\quad \text{for} \quad t > 0.
\end{aligned}
\tag{1.6}
$$

It will be assumed that y_0 is such that f is continuous (including the point $t=0$), bounded and such that $\lim_{t\to-\infty} f(t)$ exists. Then $\hat{f}(t) \in C_b^{\lim}(-\infty, 0]$ can be defined by

$$\hat{f}(t)(s) = f(t+s). \tag{1.7}$$

This permits to rewrite the equation $\dot{y}_t = F(y^t)$ in the form

$$\dot{\hat{y}} = \hat{F}(\hat{y}) + \hat{f}. \tag{1.8}$$

In this paper this approach will be applied to equations of the form

$$
\begin{aligned}
u_{tt} &= g(u_{xt}, u_x^t)_x, \qquad x \in [-1, 1], \\
g(u_{xt}, u_x^t) &= f_{\pm 1}(t) \qquad \text{at} \quad x = \pm 1.
\end{aligned}
\tag{1.9}
$$

It will be shown that if $\partial g/\partial u_{xt} > 0$, the Sobolevskii theory of quasilinear parabolic equations can be applied to this problem to prove an existence theorem locally in time.

In Section 3, we will show how the problem of stretching thin filaments of polymeric liquids reduces to an equation of the type (1.9). For this, we use a formal approximation of a similar nature as those used in rod theory [7]. Other problems that our result would apply to are the exact one-dimensional motions considered by Coleman, Gurtin and Herrera [1].

Remarks. (1) A little bit of care is needed when proceeding to (1.8). Namely, we must make sure that the abstract solution $\hat{y}(t)(s)$ actually corresponds to a function $y(t+s)$, in other words, $T_s\hat{y}(t) = \hat{y}(t+s)$. For this, observe that if $\hat{y}(t)$ is a solution of (1.8), then $\hat{z}(t) = T_s\hat{y}(t)$ solves the equation

$$\dot{\hat{z}} = \hat{F}(\hat{z}) + T_s\hat{f}. \tag{1.10}$$

So does $\hat{z}'(t) = \hat{y}(t+s)$. Hence, if you have a uniqueness theorem for (1.10), you are done. Such a uniqueness theorem will automatically come out of the considerations leading to our existence result.

(2) The approach sketched here seems to require rather restrictive conditions on the given history y_0, e.g., that y_0 and dy_0/dt are in

$C_b^{\lim}(-\infty, 0]$. However, we can circumvent such restrictions by a simple trick. Let X be a Banach space of functions on $(-\infty, 0]$ such that:

(i) $C_b^{\lim}(-\infty, 0]$ is continuously (not necessarily densely) embedded in X.

(ii) The mapping $\hat{y} \to \hat{y}(0)$ is continuous from X into $\mathbb{R}$.

(iii) For $\hat{y} \in X$ with $\hat{y}(0) = 0$, and $t > 0$, we define $S_t \hat{y}(s) = \hat{y}(s+t)$ for $s \leqslant -t$ and 0 for $s > -t$. Then $S_t \hat{y} \in X$ for any t, and $S_t \hat{y}$ depends continuously on t.

These conditions are similar to those in [12–14]. Condition (i) is a little bit more restrictive, but it includes the spaces given as applications in those papers.

Let the functional F be defined on X, and let the history y_0 be given in X. Now, let z_0 be any function such that z_0, dz_0/dt are in $C_b^{\lim}$ and $z_0(0) = y_0(0)$, $\dot{z}_0(0) = F(y_0)$. Let x_0 denote $y_0 - z_0$. Then we put

$$\begin{aligned} G(\hat{z}, t) &= F(\hat{z} + S_t x_0) \qquad \text{for} \quad t > 0 \\ &= F(\hat{z} + \hat{x}(t)) \qquad \text{for} \quad t < 0, \end{aligned}$$

where $\hat{x}(t)$ is a continuous curve in X such that $\hat{x}(0) = x_0$, $\hat{x}(t) = 0$ for $t < -T_0$. G is a functional defined on $C_b^{\lim} \times \mathbb{R}$; and G is as smooth with respect to $\hat{z}$ as F is. Moreover, G is continuous with respect to t. The equation $\dot{y} = F(y^t)$ may now be rewritten as the equivalent problem $\dot{z} = G(z^t, t)$, which has an initial history in $C_b^{\lim}$. The problem discussed in Section 2 can be approached in the same spirit.

The reader might ask why we choose this method rather than trying to represent the problem as an evolution equation in X. The reason for this is the following: if F is a continuous functional on $C_b^{\lim}$, and the history of y lies in $C_b^{\lim}$, then the history of $F(y^t)$ also lies in $C_b^{\lim}$, in other words, $\hat{F}$ (as defined above) is an operator in $C_b^{\lim}$. In X, this property will in general not hold and may require severe restrictions on admissible nonlinearities.

2. The Main Result

We study the following problem:

$$\begin{aligned} u_{tt} &= g(u_{xt}, u_x^t)_x \qquad \text{for} \quad x \in [-1, 1], \\ g(u_{xt}, u_x^t) &= f_{\pm 1}(t) \qquad \text{at} \quad x = \pm 1. \end{aligned} \tag{2.1}$$

In view of the application we have in mind, we require g to be defined only on an open subset of the history space $C_b^{\lim}(-\infty, 0]$. We denote this open subset by Ω. We assume then

(i) g is a smooth function from $\mathbb{R} \times \Omega$ into $\mathbb{R}$. Moreover, there exists $\varepsilon > 0$ such that $D_1 g > \varepsilon$ on $\mathbb{R} \times \Omega$. (By $D_1 g$, $D_2 g$ we denote the Fréchet derivatives of g w.r. to the first and second argument.)

Let us first study the boundary conditions. They yield equations of the form

$$g(\dot{y}, y^t) = f(t) \tag{2.2}$$

(we have substituted y for $u_x(\pm 1)$). According to our assumptions on g, we can resolve this equation w.r. to $\dot{y}$, leading to an equation of the form

$$\dot{y} = h(y^t, f(t)), \tag{2.3}$$

where h is smooth from $\Omega \times \mathbb{R}$ into $\mathbb{R}$.

As usual, we study (2.2) for $t > 0$ with an initial condition $y(t) = y_0(t)$ for $t < 0$. On this initial condition we assume

(ii) $y_0 \in \Omega$ has a continuous bounded derivative, and $\lim_{s \to -\infty}(dy_0(s)/ds) = 0$. Moreover, the compatibility condition

$$g\left(\frac{dy_0}{dt}(0), y_0\right) = f(0)$$

is satisfied.

We now define $f(t)$ by

$$\begin{aligned} f(t) &= g\left(\frac{dy_0(t)}{dt}, T_t y_0\right) && \text{for} \quad t < 0 \\ &= \text{the given } f && \text{for} \quad t \geqslant 0. \end{aligned} \tag{2.4}$$

As in Section 1, let $\hat{y}(t)(s) = y(t+s)$, $\hat{f}(t)(s) = f(t+s)$. Then (2.3) is rewritten in the abstract form

$$\dot{\hat{y}} = \hat{h}(\hat{y}, \hat{f}(t)), \tag{2.5}$$

where $\hat{h}$ is smooth from $\Omega \times C_b^{\lim}(-\infty, 0]$ into $C_b^{\lim}(-\infty, 0]$. The Picard–Lindelöf theorem now immediately yields the following result.

THEOREM 2.1. *Let* (i), (ii) *hold. Then, for some* $T > 0$, *Eq.* (2.5) *with the initial condition* $\hat{y}(0) = y_0$ *has a unique solution* $\hat{y} \in C^1([0, T], C_b^{\lim}(-\infty, 0])$.

We now turn to the study of (2.1). It is convenient to rewrite this equation as a system. We put $u_x = p$, $u_t = \tilde{r}$, $u_{xx} = q$. Then (2.1) is equivalent to

$$\begin{aligned} \dot{p} &= \tilde{r}_x, \\ \dot{q} &= \tilde{r}_{xx}, \\ \dot{\tilde{r}} &= D_1 g(\tilde{r}_x, p^t) \cdot \tilde{r}_{xx} + D_2 g(\tilde{r}_x, p^t) \cdot q^t. \end{aligned} \tag{2.6}$$

Remark. We have assumed here that the pointwise derivative $q = p_x$ is actually the derivative in the sense to the space $C_b^{\text{lim}}(-\infty, 0]$. This is justified if $q(t+s, x)$ is uniformly bounded and uniformly continuous for $x \in [-1, 1]$, $s \in (-\infty, 0]$. Our assumptions on the initial history will be such that this is implied.

From our study of the boundary problem, we know solutions $u_x(t, 1) = y_1(t)$, $u_x(t, -1) = y_{-1}(t)$. With (2.6) we must then associative the boundary conditions

$$\tilde{r}_x(t, 1) = \frac{dy_1(t)}{\partial t}, \qquad \tilde{r}_x(t, -1) = \frac{dy_{-1}(t)}{\partial t}. \tag{2.7}$$

Letting

$$r = \tilde{r} - \frac{1}{2} x \left(\frac{dy_1(t)}{\partial t} + \frac{dy_{-1}(t)}{\partial t} \right) - \frac{1}{4} x^2 \left(\frac{dy_1(t)}{\partial t} - \frac{dy_{-1}(t)}{\partial t} \right),$$

we obtain homogeneous Neumann conditions for r.

Before we reformulate (2.6) as an abstract evolution problem, we need to introduce some function spaces. Let H^n denote the space of all real-valued functions on $[-1, 1]$, which have n square integrable derivatives. By $C_b^{\text{lim}}((-\infty, 0], H^n)$ we denote the space of continuous, bounded functions from $[-\infty, 0]$ into H^n, which converge to a limit at $-\infty$. We assume

(iii) g (regarded as acting pointwise w.r. to the space variable) is a smooth mapping from $\mathbb{R} \times \tilde{\Omega}$ into H^1, where $\tilde{\Omega}$ is an open subset of $C_b^{\text{lim}}((-\infty, 0], H^1)$. On $\mathbb{R} \times \tilde{\Omega}$, we have $D_1 g \geqslant \varepsilon > 0$.

We are studying an initial history value problem, i.e., u is given for $t \leqslant 0$. We assume that this initial history is such that

(iv) u_t lies in $C_b^{\text{lim}}((-\infty, 0], H^3)$, and u_{tt} lies in $C_b^{\text{lim}}((-\infty, 0], H^1)$. Moreover, u_x lies in $C_b^{\text{lim}}((-\infty, 0)], H^2 \cap \tilde{\Omega})$.

The history of u provides histories for p, q and r in the obvious way. We make sure that the history satisfies (2.6) by adding an inhomogeneous term $f^*(t, x)$ to the right side of the third equation. We assume a compatitility condition

(v) The initial history is such that $f^*(0, x) = 0$.

Finally, we assume the following smoothness conditions:

(vi) f^* (regarded as valued in H^1) is uniformly Hölder continuous in t. Moreover, y_1 and y_{-1} (the boundary values of u_x) are in C^2, and d^2y_1/dt^2, d^2y_{-1}/dt^2 are uniformly Hölder continuous.

For abbreviation, let us put

$$\varphi(t, x) = \frac{1}{2} x \left(\frac{dy_1(t)}{dt} + \frac{dy_{-1}(t)}{dt}\right) - \frac{1}{4} x^2 \left(\frac{dy_1(t)}{dt} - \frac{dy_{-1}(t)}{dt}\right),$$

$$\varphi_1(t, x) = \frac{\partial\varphi(t, x)}{\partial x}, \qquad \varphi_2(t, x) = \frac{\partial^2\varphi(t, x)}{\partial x^2}, \qquad \varphi_3(t, x) = \frac{\partial\varphi(t, x)}{\partial t}.$$

As before, we define $\hat{p}(t)(s) = p(t-s)$, etc. Equation (2.6) now assumes the following form:

$$\begin{aligned} \dot{\hat{p}} &= \hat{r}_x + \hat{\varphi}_1, \\ \dot{\hat{q}} &= \hat{r}_{xx} + \hat{\varphi}_2, \\ \dot{\hat{r}} &= D_1\hat{g}(\hat{r}_x + \hat{\varphi}_1, \hat{p})(\hat{r}_{xx} + \hat{\varphi}_2) + D_2\hat{g}(\hat{r}_x + \hat{\varphi}_1, \hat{p})\hat{q} - \hat{\varphi}_3 + \hat{f}^* \end{aligned} \tag{2.8}$$

with the boundary condition $\hat{r}_x = 0$ at $x = \pm 1$.

We quote the following result of Sobolevskii [10].

THEOREM 2.2. *Consider an evolution equation*

$$\dot{v} + A(t, v)v = f(t, v) \tag{2.9}$$

in a Banach space E. Assume the following are true: $A_0 = A(0, v_0)$ *is a linear operator whose domain D is dense in E. Let there be* $d \in \mathbb{R}$ *such that, for* $\operatorname{Re}\lambda \geqslant d$, *the operator* $A_0 + \lambda I$ *has a bounded inverse, and*

$$\|(A_0 + \lambda I)^{-1}\| \leqslant C(|\lambda| + 1)^{-1}. \tag{2.10}$$

Let $\tilde{A}_0$ *denote* $A_0 + dI$. *For some* $\alpha \in [0, 1)$ *and for any* $v \in E$, $\|v\| \leqslant R$, *let the operator* $A(t, \tilde{A}_0^{-\alpha}v)$ *be defined in D and*

$$\begin{aligned} &\|[A(t, \tilde{A}_0^{-\alpha}v) - A(\tau, \tilde{A}_0^{-\alpha}w)]\,\tilde{A}_0^{-1}\| \\ &\qquad \leqslant C(R)[|t-\tau|^{\varepsilon} + \|v - w\|], \qquad \varepsilon \in [0, 1] \end{aligned} \tag{2.11}$$

for any $0 \leqslant t$, $\tau \leqslant T$, $\|v\| \leqslant R$, $\|w\| \leqslant R$. *In this region, let*

$$\|f(t, \tilde{A}_0^{-\alpha}v) - f(\tau, \tilde{A}_0^{-\alpha}w)\| \leqslant C(R)[|t-\tau|^{\varepsilon} + \|v - w\|]. \tag{2.12}$$

Finally, for some $\beta > \alpha$, *let* $v_0 \in D(\tilde{A}_0^{\beta})$, *and let* $\|\tilde{A}_0^{\alpha}v_0\| < R$. *Then there exists one and only one solution of Eq.* (2.9) *which is defined on an interval*

$[0, t_0]$, *is continuous for* $t \in [0, t_0)$, *is continuously differentiable for* $t > 0$, *and satisfies the initial condition*

$$v(0) = v_0. \tag{2.13}$$

We apply this to (2.8) by making the following identifications. Let E denote the space $C_b^{\text{lim}}((-\infty, 0], H^1)^3$ and let $v = (\hat{p} - \hat{p}_0, \hat{q} - \hat{q}_0, \hat{r} - \hat{r}_0)$, where $\hat{p}_0$, $\hat{q}_0$, $\hat{r}_0$ denote the given initial conditions. The operator $A(t, v)$ is defined by $-A(t, v)(\hat{p}', \hat{q}', \hat{r}') = (\hat{r}'_x, \hat{r}'_{xx}, D_1 \hat{g}(\hat{r}_x + \hat{\varphi}_1, \hat{p})\, \hat{r}'_{xx})$ incorporating the boundary condition $\hat{r}'_x = 0$ and $f(t, v)$ is identified with the remainder of the right-hand side of (2.8). Then the initial condition $v_0 = 0$ is in $D(A_0)$, i.e., we can choose $\beta = 1$. Moreover, for appropriate choice of R, it is clear that (2.11), (2.12) hold with $\alpha = \frac{1}{2}$, and ε the Hölder exponent of condition (vi). Condition (2.10) finally follows from the positivity of $D_1 \hat{g}$ and the fact that, for positive $\varphi(x)$, the operator $r \mapsto \varphi(x)\, r_{xx}$ with homogeneous Neumann conditions is selfadjoint in H^1, provided that H^1 is equipped with the scalar product

$$((p, q)) = \int_{-1}^{1} \frac{1}{\varphi(x)}\, p(x)\, q(x)\, dx + \int_{-1}^{1} p_x(x)\, q_x(x)\, dx.$$

This product is equivalent to the usual one.

We thus obtain

THEOREM 2.3. *Let hypotheses* (i)–(vi) *hold. Then, for some* $T > 0$, *Eq.* (2.8) *has a unique solution in* $C^1([0, T], C_b^{\text{lim}}(-\infty, 0], H^1)^3)$, *which satisfies the given initial conditions.*

Remarks. Certain models for viscoelastic liquids express the stress as a functional not only of the history of strain, but also of the history of the rate of strain (see, e.g., [11]). In this case we will have to deal with equations of the form

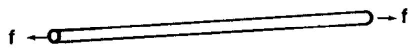
$$u_{tt} = g(u_{xt}, u_x^t, u_{xt}^t)_x. \tag{2.14}$$

If $D_3 g$ is small compared to $D_1 g$, it can be shown by perturbation arguments that the asumptions necessary for the theorems above remain valid. In many cases one has to deal with functionals that, although they involve the rate of strain, also have a smoothing property. The stress is (at least) as often differentiable w.r. to time as the strain, and no derivative is lost. For example, take a constitutive law of the form $\sigma(t) = \int_{-\infty}^{t} a(t-s) f(u_{xt}(s))\, ds$. This leads to $\sigma_t(t) = \int_{-\infty}^{t} a'(t-s) f(u_{xt}(s))\, ds + a(0) f(u_{xt}(t))$. Thus σ_t is given by an expression involving only u_{xt} and not u_{xtt}. If the history of u_{xt} enters (2.4) via such expressions, it is possible to apply the same methods as before to the time differentiated version of (2.4).

The terms involving histories are then still of "lower order" and can be included in the "f-part." We leave the details to the reader.

3. The Stretching of Thin Filaments of Viscoelastic Liquids

We consider the problem of stretching a thin filament of a viscoelastic liquid by a force f applied to its ends:



We shall derive an equation describing this problem in an analogous fashion as I did in [8] assuming a special constitutive law. We start with the general three-dimensional equations of continuum mechanics and then introduce a one-dimensional approximation analogous to rod theory [7].

Let $(\zeta^1, \zeta^2, \zeta^3)$ denote "body coordinates," i.e., coordinates labelling a specific particle in the fluid. These coordinates can be identified with the position of the particle in space, when the fluid is in a certain "reference state." This reference state will be chosen such that the filament is cylindrical in this state. By (y^1, y^2, y^3) we denote Cartesian coordinates in space. We are interested in finding trajectories of particles, i.e., a functional dependence $y(\zeta^1, \zeta^2, \zeta^3, t)$.

The deformation gradient F is defined by $F^i_j = \partial y^i / \partial \zeta^j$ and the right Cauchy–Green strain tensor is defined by

$$C = F^T F. \tag{3.1}$$

Following Lodge [3], we introduce a Lagrangian stress tensor S, which is related to the Cauchy stress tensor T by

$$S = F^{-1} T (F^T)^{-1}. \tag{3.2}$$

S is related to C by a constitutive law. We assume that this constitutive equation consists of three terms: a pressure term, a Newtonian viscosity and a "genuine" polymeric contribution, which is a smooth functional of the strain history; i.e., we have

$$S = -pC^{-1} - \eta \frac{\partial}{\partial t} C^{-1} + \mathscr{F}(C^t). \tag{3.3}$$

The incompressibility condition is expressed by

$$\det C = 1. \tag{3.4}$$

The equation of motion is then given by

$$\rho y_{tt} = \mathrm{Div}(FS), \tag{3.5}$$

where Div refers to the body coordinate system, or, in coordinates

$$\rho y_{tt}^k = \sum_{r,s} \frac{\partial y^k}{\partial \zeta^r} \frac{\partial}{\partial \zeta^s} S^{rs} + \frac{\partial^2 y^k}{\partial \zeta^r \partial \zeta^s} S^{rs}. \tag{3.6}$$

We have to supplement (3.6) by the boundary conditions. Let f denote the force acting on a surface divided by the surface area in the *reference* state. f is referred to space coordinates. Then the boundary condition reads

$$FSn = f, \tag{3.7}$$

where n denotes the unit normal of the surface in the body coordinate system.

We now specialize our considerations to the filament. Coordinates are chosen such that ζ^1 and y^1 are in the direction of the filament, and the other coordinates are transverse. It is assumed that in the reference state the filament is axisymmetric and cylindrical, i.e., ζ^1 ranges over a finite interval, say $[-1, 1]$, and $r = \sqrt{(\zeta^2)^2 + (\zeta^3)^2}$ ranges from 0 to δ, the radius of the filament. We shall assume that δ is small. In a formal argument, we shall derive an equation for the terms of highest order w.r. to δ. The argument is analogous to the one given by Nariboli [7] for the problem of longitudinal elastic waves in a thin rod.

We assume that the solutions y^1, y^2, y^3 and the pressure p are such that they depend smoothly on r, and that the coefficients in a Taylor expansion w.r. to powers of r are smooth functions of δ. Let us put $\hat{\zeta}^2 = \zeta^2/\delta$, $\hat{\zeta}^3 = \zeta^3/\delta$, $\hat{r} = r/\delta$, so that the lateral surface corresponds to $\hat{r} = 1$. For the solutions we now make the Ansatz:

$$\begin{aligned}
y^1(\zeta^1, \delta\hat{\zeta}^2, \delta\hat{\zeta}^3) &= \sum_{\nu=0}^{N} \delta^{2\nu} P_\nu(\zeta^1, \hat{r}^2) + \mathcal{O}(\delta^{2N+2}),\\
y^2(\zeta^1, \delta\hat{\zeta}^2, \delta\hat{\zeta}^3) &= \delta\hat{\zeta}^2 \sum_{\nu=0}^{N} \delta^{2\nu} Q_\nu(\zeta^1, \hat{r}^2) + \mathcal{O}(\delta^{2N+3}),\\
y^3(\zeta^1, \delta\hat{\zeta}^2, \delta\hat{\zeta}^3) &= \delta\hat{\zeta}^3 \sum_{\nu=0}^{N} \delta^{2\nu} Q_\nu(\zeta^1, \hat{r}^2) + \mathcal{O}(\delta^{2N+3}),\\
p(\zeta^1, \delta\hat{\zeta}^2, \delta\hat{\zeta}^3) &= \sum_{\nu=0}^{N} \delta^{2\nu} R_\nu(\zeta^1, \hat{r}^2) + \mathcal{O}(\delta^{2N+2}),
\end{aligned} \tag{3.8}$$

where P_ν, Q_ν, R_ν are polynomials of νth degree in $\hat{r}^2$. We insert this Ansatz into (3.4), (3.6) and the boundary conditions (3.7) on the lateral surface of the filament. By retaining only lowest-order terms in δ, we shall obtain an equation for $P_0(\zeta^1)$. In the following analysis we shall carry out the series expansions (3.8) only so far as is necessary for this purpose.

For the Cauchy–Green tensor we find

$$C=\begin{pmatrix} \left(\frac{\partial P_0}{\partial \zeta^1}\right)^2 & \delta\zeta^2\cdot\varphi & \delta\zeta^3\cdot\varphi \\ \delta\zeta^2\cdot\varphi & Q_0^2 & 0 \\ \delta\zeta^3\cdot\varphi & 0 & Q_0^2 \end{pmatrix}+\mathcal{O}(\delta^2), \tag{3.9}$$

where

$$\varphi=\frac{\partial P_0}{\partial \zeta^1}\cdot\frac{\partial P_2}{\partial \hat{r}}\hat{r}^{-1}+\frac{\partial Q_0}{\partial \zeta^1}Q_0.$$

Thus, in the lowest order, the incompressibility condition (3.4) yields

$$\left(\frac{\partial P_0}{\partial \zeta^1}\right)^2 Q_0^4=1. \tag{3.10}$$

Next, let us consider the boundary conditions on the lateral surface. At a boundary point where $\zeta^3=0$ these yield the equations $S^{21}=S^{22}=S^{23}=0$. (Because of the radial symmetry it suffices to consider these boundary points; if the traction vanishes there, it does so everywhere else.) S^{23} vanishes identically as a result of the radial symmetry. For S^{22} we obtain the following terms of order $O(1)$:

$$S^{22}=-R_0Q_0^{-2}-\eta\frac{\partial}{\partial t}(Q_0^{-2})+\mathcal{F}^{22}(C_0^t), \tag{3.11}$$

where C_0 denotes the zeroth order term in C. Moreover, we have to satisfy

$$S^{21}=0. \tag{3.12}$$

It follows from material isotropy that, if C is diagonal, then S must be diagonal, too (see, e.g., [3]). Hence (3.12) is satisfied in $\mathcal{O}(\delta)$, if $\varphi=0$. The equation of motion now yields the following equation for P_0:

$$\begin{aligned}\rho\frac{\partial^2 P_0}{\partial t^2}=\frac{\partial P_0}{\partial \zeta^1}&\left\{\frac{\partial}{\partial \zeta^1}\left(-R_0\left(\frac{\partial P_0}{\partial \zeta^1}\right)^{-2}-\eta\frac{\partial}{\partial t}\left(\frac{\partial P_0}{\partial \zeta^1}\right)^{-2}+\mathcal{F}^{11}(C_0^t)\right)\right\}\\ &+\frac{\partial^2 P_0}{(\partial \zeta^1)^2}\cdot\left(-R_0\left(\frac{\partial P_0}{\partial \zeta^1}\right)^{-2}-\eta\frac{\partial}{\partial t}\left(\frac{\partial P_0}{\partial \zeta^1}\right)^{-2}+\mathcal{F}^{11}(C_0^t)\right).\end{aligned} \tag{3.13}$$

In order to simplify notation, let us write u for P_0 and x for ζ^1. Since C_0 depends only on u_x, we shall also write $\mathscr{F}^{11}(C_0^t)=\mathscr{F}^1(u_x^t)$, $\mathscr{F}^{22}(C_0^t)=\mathscr{F}^2(u_x^t)$. When we substitute R_0 from (3.11), we obtain the following equation from (3.13)

$$\rho u_{tt}=3\eta\frac{\partial^2}{\partial x\,\partial t}\left(-\frac{1}{u_x}\right)+\frac{\partial}{\partial x}\{u_x\mathscr{F}^1(u_x^t)-u_x^{-2}\mathscr{F}^2(u_x^t)\}. \tag{3.14}$$

Finally, we have to derive the boundary conditions at the ends of the filament. As noted in [7], expansions of the type (3.8) cannot be used to satisfy these in every order in δ, and a "boundary layer" arises. However, for the first-order approximation, we need not deal with this. When we insert (3.8) into the boundary conditions at the ends, then all traction components not in the direction of the filament are $\mathscr{O}(\delta)$. For the longitudinal traction, we find in order $\mathscr{O}(1)$:

$$3\eta\frac{\partial}{\partial t}\left(-\frac{1}{u_x}\right)+\{u_x\mathscr{F}^1(u_x^t)-u_x^{-2}\mathscr{F}^2(u_x^t)\}=f(t), \tag{3.15}$$

where $f(t)$ denotes the force acting on the end, divided by the cross-sectional area in the reference state, $\pi\delta^2$.

Remark. Our analysis in Section 2 breaks down if $\eta=0$. However, we have seen from numerical studies [5, 6] of a specific model that, for certain fluids (like molten polyethylene), η must be assumed very large in order to have significant influence on the solution. Experimental values for η are not available. Thus a theory that includes the case $\eta=0$ seems to have more than academic interest. The equations are not parabolic in this case, but can for a wide class of constitutive laws be classified as hyperbolic. This will be discussed in a forthcoming paper [9]. In a restricted situation, Lodge, McLeod and Nohel [4] have discussed solutions to the boundary equation (3.15) in the limit $\eta\to 0$.

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